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Construction of alternative Hamiltonian structures for field equations

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Abstract

We use symmetry vectors of nonlinear field equations to build alternative Hamiltonian structures. We construct such structures even for equations which are usually believed to be non-Hamiltonian such as heat, Burger and potential Burger equations. We improve on a previous version of the approach using recursion operators to increase the rank of the Poisson bracket matrices. Cole–Hopf and Miura-type transformations allow the mapping of these structures from one equation to another.

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1. Introduction

The standard way of constructing Hamiltonian theories starts from a Lagrangian by defining the momenta and the Hamiltonian. The procedure is, of course, well known and may be found in many textbooks. Nevertheless, it is interesting to attempt the construction of Hamiltonian structures for systems of differential equations, without recourse to a Lagrangian which may be either unknown or may even fail to exist, starting from just the equations of motion. One of us [1, 2] devised a general technique for the construction of Hamiltonian structures using symmetries and constants of motion of dynamical systems (see also [3]). In this paper, we enrich this technique using in addition to recursion operators Cole–Hopf and Miura-type transformations in order to raise the rank of the Poisson-bracket operators and to relate the structures of different nonlinear equations, respectively.

2. The Hamiltonian structure

Consider a system of nonlinear differential equations

$$u_t^{(\alpha)}[x^{(a)}, t] = f^{(\alpha)}[u[x^b, t]] \tag{1}$$

where the Greek indices label the dependent variables (or fields) while the Latin indices are used to denote the independent variables and $u_t \equiv \partial u/\partial t$. The field vector $f^{(\alpha)}[u[x^b, t]]$ depends implicitly on time, t, through the dependent variables.

A Hamiltonian structure for (1) consists of an antisymmetric operator, $J = J[x^{(a)}, x^{(b)}]$ and a Hamiltonian H such that $J = J[x^{(a)}, x^{(b)}]$ is the Poisson bracket for the dynamical variables, which are in general non-canonical

$$\{\mathcal{F}, \mathcal{G}\} = \int \delta\mathcal{F} J \delta\mathcal{G}\, \mathrm{d}\boldsymbol{x} \tag{2}$$

where $\mathcal{F}$ and $\mathcal{G}$ are functional on the field's variables and δ is the variational derivative of the functionals. Additionally, $J[x^{(a)}, x^{(b)}]$ must be antisymmetric:

$$\{\mathcal{F}, \mathcal{G}\} = -\{\mathcal{G}, \mathcal{F}\} \tag{3}$$

and must satisfy the Jacobi identity

$$\{\mathcal{R}, \{\mathcal{F}, \mathcal{G}\}\} + \{\mathcal{F}, \{\mathcal{G}, \mathcal{R}\}\} + \{\mathcal{G}, \{\mathcal{R}, \mathcal{F}\}\} \equiv 0 \tag{4}$$

and must reproduce, in conjunction with the Hamiltonian H, the dynamical equation (1), i.e.,

$$\int J\delta H = f. \tag{5}$$

It has been proved [1,2] that one solution to the problem of finding a Hamiltonian structure for a given dynamical system is provided by one constant of the motion which may be used as the Hamiltonian H, and a symmetry vector η which allows for the construction of a Poisson-bracket operator J. The constant of motion and the symmetry vector satisfy

$$\mathcal{L}_f H = 0 \tag{6}$$

$$(\partial_t + \mathcal{L}_f)\eta = 0 \tag{7}$$

respectively, where $\mathcal{L}_f$ is the Lie derivative along f. In addition, it is required that the deformation K of H along η

$$K \equiv \mathcal{L}_\eta H \tag{8}$$

is non-vanishing.

To simplify the notation we use only one dependent variable and one independent variable. The extension to a larger number of variables is straightforward. The Poisson-bracket operator $J[x, y]$ is constructed as the antisymmetrized product of the flow vector $f[x]$ and the 'normalized' symmetry vector η/K

$$J[x, y] = \frac{1}{K}(f[x]\eta[y] - f[y]\eta[x]). \tag{9}$$

The Poisson-bracket matrix constructed thus has rank 2 and is, therefore, singular. Adding together two Poisson-bracket matrices constructed according to (9) will not increase the rank but will just redefine the symmetry vector used. A method of increasing the rank of such a Poisson bracket matrix is presented in [1]. In summary, the technique for increasing the rank of the Poisson-bracket matrix consists of the following steps: consider two new symmetries η_2 and η_3 such that they satisfy the following conditions:

$$\partial_t \eta_2 = \partial_t \eta_3 = 0 \tag{10}$$

$$\mathcal{L}_{\eta_2} H = \mathcal{L}_{\eta_3} H = 0 \tag{11}$$

$$\mathcal{L}_{\eta_2}\eta_3 = \mathcal{L}_\eta \eta_2 = \mathcal{L}_\eta \eta_3 = 0. \tag{12}$$

The new Poisson-bracket matrix, $\tilde{J}[x, y]$, is then defined by

$$\tilde{J}[x, y] = J[x, y] + \eta_2[x]\eta_3[y] - \eta_2[y]\eta_3[x]. \tag{13}$$

This procedure can be repeated at will, producing an increase of two units in the rank of the matrix each time it is performed.

3. The heat equation

To apply the procedure described in the previous section, we use the heat equation with a periodic boundary condition. The heat equation for $\omega[x, t]$ subject to periodic boundary conditions is

$$\begin{aligned} &\omega_t[x, t] = \omega_{xx}[x, t] \\ &\omega[0, t] = \omega[a, t] \end{aligned} \tag{14}$$

and the function $\omega[x, t]$ should also satisfy the initial condition $\omega[x, 0] = f(x)$. Equation (14) implies that all even derivatives of the field $\omega[x, t]$ are also subject to periodic boundary conditions, i.e., $\omega^{(2n)}[0, t] = \omega^{(2n)}[a, t]$ with $n = 0, 1, 2, \ldots$.

The equation (14) has an integral of motion

$$H = \int_0^a \omega[x, t]\,\mathrm{d}x. \tag{15}$$

One symmetry transformation for this equation is

$$\eta^{(0)} = \alpha[x, t]\partial_\omega \tag{16}$$

where $\alpha[x, t]$ is any solution of the equation (14). Take, in particular, $\alpha[x, t] = 1$.

The integral of motion (15) has a non-vanishing deformation along $\eta^{(0)}$

$$\mathcal{L}_{\eta^{(0)}} H = \int_0^a \frac{\delta H}{\delta\omega[x, t]}\eta^{(0)}\,\mathrm{d}x = a. \tag{17}$$

Following the above procedure we can construct a Poisson bracket structure for (14) (see [3])

$$J^{(0)}[x, y] = \frac{\omega_{xx}[x, t] - \omega_{yy}[y, t]}{a}. \tag{18}$$

Another symmetry transformation for equation (14) is a dilatation symmetry defined by

$$\eta^{(D)} = (b\omega[x, t] - 2t\omega_{xx}[x, t] - x\omega_x[x, t])\partial_\omega \tag{19}$$

where b is any real number. This symmetry transformation satisfies the conditions outlined in section 2, for $b \neq -1$. In particular, the deformation of (15) along $\eta^{(D)}$ is

$$\mathcal{L}_{\eta^{(D)}} H = \int_0^a \frac{\delta H}{\delta\omega[x, t]}\eta^{(D)}\,\mathrm{d}x = (1 + b)H. \tag{20}$$

We may now construct a second Poisson-bracket structure using this symmetry

$$J^{(D)}[x, y] = \frac{\omega_{xx}[x, t]\eta^{(D)}[y, t] - \omega_{yy}[y, t]\eta^{(D)}[x, t]}{(1 + b)H}. \tag{21}$$

Another symmetry vector may be used for constructing a Poisson bracket structure, as we will see in the next section, namely

$$\eta^{(2)} = \frac{\epsilon}{2}\left(t + \frac{1}{2}x^2\right). \tag{22}$$

The corresponding Poisson-bracket structure is

$$J^{(2)}[x, y] = \frac{\omega_{xx}[x, t]\eta^{(2)}[y, t] - \omega_{yy}[y, t]\eta^{(2)}[x, t]}{\int_0^a \eta^{(2)}[x, t]\,\mathrm{d}x}. \tag{23}$$

4. The recursion operators

The recursion operators [4–6] for the heat equation are

$$\Lambda_1[\omega] = D \tag{24}$$

$$\Lambda_2[\omega] = tD + \frac{x}{2} \tag{25}$$

$$\Lambda_1^{-1}[\omega] = D^{-1} \tag{26}$$

$$\Lambda_2^{-1} = \frac{\exp(-\frac{x^2}{4t})}{\sqrt{t}} D^{-1} \frac{\exp(\frac{x^2}{4t})}{\sqrt{t}}. \tag{27}$$

These operators satisfy the following equation (see [4, 5, 7])

$$(\partial_t + \mathcal{L}_f)\Lambda = \partial_t\Lambda + \Lambda'[f] - f'[\Lambda] + \Lambda[f'] = 0 \tag{28}$$

where the prime indicates the Frechet derivative.

Now we apply the operator (25) to the constant symmetry ϵ and obtain a family of symmetries. The first four of them are

$$\eta^{(0)} = (\Lambda_2)^0\,\epsilon = \epsilon \tag{29}$$

$$\eta^{(1)} = (\Lambda_2)^1\,\epsilon = \tfrac{1}{2}\epsilon x \tag{30}$$

$$\eta^{(2)} = (\Lambda_2)^2\,\epsilon = \frac{\epsilon}{2}\left(t + \frac{x^2}{2}\right) \tag{31}$$

$$\eta^{(3)} = (\Lambda_2)^3\,\epsilon = \frac{\epsilon}{2}\left(3tx + \frac{x^3}{4}\right). \tag{32}$$

Notice that the symmetries obtained, are also special solutions of the heat equation called the ‘heat polynomial’ [8] that are invariant with respect to the action of dilatation symmetry. Taking into account the equation of motion (14) and the integral of motion (15), it is a straightforward matter to prove that

$$\mathcal{L}_{(\Lambda_2)^n\epsilon} H = \int_0^a (\Lambda_2)^n\epsilon\,\mathrm{d}x = K \tag{33}$$

where $K = 0$ if n is odd, and $K \neq 0$ if n is even. On the other hand, we have

$$\mathcal{L}_{(\Lambda_2)^n\epsilon}(\Lambda_1)^m(\omega_x) = -D^{(m+1)}((\Lambda_2)^n\epsilon) = 0 \Leftrightarrow m \geqslant n. \tag{34}$$

From this relationship we get

$$\mathcal{L}_f((\Lambda_2)^2\epsilon) = -\frac{\epsilon}{2}. \tag{35}$$

Furthermore, one may get the Jacobi identity (4) for the Poisson-bracket operator (9) by using (35) to show that, in fact

$$J[x, y]\mathcal{L}_f\eta[z] + J[y, z]\mathcal{L}_f\eta[x] + J[z, x]\mathcal{L}_f\eta[y] \equiv 0. \tag{36}$$

Therefore, we may use $(\Lambda_2)^2\epsilon$ for the construction of a Hamiltonian structure for equation (14) as in (23).

5. Raising the rank of Poisson-bracket operators

The symmetries $(\Lambda_1)^{2n}(\omega_x)$ with $m, n = 0, 1, 2\ldots$ satisfy the following equations:

$$\partial_t(\Lambda_1)^{2n}(\omega_x) = 0 \tag{37}$$

$$\mathcal{L}_{(\Lambda_1)^{2n}(\omega_x)} H = 0 \tag{38}$$

$$\mathcal{L}_{(\Lambda_1)^{2n}(\omega_x)}(\Lambda_1)^{2m}(\omega_x) = 0 \tag{39}$$

$$\mathcal{L}_{(\Lambda_1)^{2n}(\omega_x)}\epsilon = 0 \tag{40}$$

where we use the periodic boundary conditions for even derivatives of the field in (38). Furthermore, for $2n > 2$ we have

$$\mathcal{L}_{(\Lambda_1)^{2n}(\omega_x)}(\Lambda_2)^2(\epsilon) = 0. \tag{41}$$

Keeping these properties and the conditions (10)–(12) in mind, we can raise the rank of the Poisson-bracket structures (18) and (23) by defining

$$J^{(0)}_{\infty} = J^{(0)} + \sum_{m\neq n=1}^{\infty} A_{nm}(\omega_{2n}(x)\omega_{2m}(y) - \omega_{2n}(y)\omega_{2m}(x)) \tag{42}$$

$$J^{(2)}_{\infty} = J^{(2)} + \sum_{m\neq n=2}^{\infty} B_{nm}(\omega_{2n}(x)\omega_{2m}(y) - \omega_{2n}(y)\omega_{2m}(x)) \tag{43}$$

where A_{nm} and B_{nm} are real antisymmetric matrices.

6. The Cole–Hopf transformation

We have already introduced the heat equation (14), while the potential Burger and the Burger equations are given by

$$u_t[x,t] = u_{xx}[x,t] + u_x^2[x,t] \tag{44}$$

and

$$v_t[x,t] = v_{xx}[x,t] + 2v_x^2[x,t]v[x,t], \tag{45}$$

respectively. The variables $\omega[x,t], u[x,t]$ and $v[x,t]$ are related by the following transformations [9–12]

$$\omega[x,t] = F(u[x,t]) \equiv \exp(u[x,t]) \tag{46}$$

$$v[x,t] = B(u[x,t]) \equiv u_x[x,t] \tag{47}$$

$$v[x,t] = \frac{\omega_x[x,t]}{\omega[x,t]}. \tag{48}$$

The recursion operators are mapped into

$$\Lambda[\omega] = F'\tilde{\Lambda}[u]F'^{-1} \tag{49}$$

$$\hat{\Lambda}[v] = B'\tilde{\Lambda}[u]B'^{-1} = B'F'^{-1}\Lambda[\omega]F'B'^{-1} \tag{50}$$

where $F' = \exp(u[x,t])$ and $B' = D$ are the Jacobian of the transformations (46) and (47), respectively. Thus, the recursion operators are

$$\tilde{\Lambda}_1[u] = \exp(-u[x,t])D\exp(u[x,t]) = D + u_x[x,t] \tag{51}$$

$$\tilde{\Lambda}_2[u] = tD + tu_x[x,t] + \tfrac{1}{2}x \tag{52}$$

$$\hat{\Lambda}_1[v] = Dv[x,t]D^{-1} + D \tag{53}$$

$$\hat{\Lambda}_2[v] = t(Dv[x,t]D^{-1} + D) + \tfrac{1}{2}(D^{-1} + x). \tag{54}$$

The symmetries vectors in the Burger and potential Burger equations are

$$\eta_v = B'\eta_\omega = D\exp(-u[x,t])\eta_\omega \tag{55}$$

$$\eta_u = F'\eta_\omega = \exp(-u[x,t])\eta_\omega \tag{56}$$

for any symmetry vector of the heat equation η_ω. For instance, one symmetry vector for the potential Burger equation (44) is

$$\eta_u = \alpha[x,t]\exp(-u[x,t])\partial_u \tag{57}$$

where $\alpha[x, t]$ is a solution of heat equation (14).

Applying the recursion operator $\tilde{\Lambda}_1[u]$ on (57) we have for $n = 0, 1, 2 \ldots$:

$$\tilde{\Lambda}_1^{(n)}[u]\eta_u = \alpha_{nx}[x, t] \exp(-u[x, t])\partial_u. \tag{58}$$

Note that these symmetries commute among themselves and in particular with $\alpha_{xx}[x, t] \exp(-u[x, t]) = u_{xx}[x, t] + u^2[x, t]$. Keeping in mind the general outline for the construction of Hamiltonian structures with the help of one symmetry that does not deform the Hamiltonian trivially, we get the following results for the potential Burger equation:

$$J_u[x, y] = \eta_\omega[y] \exp(-u[y, t])\omega_{xx}[x, t] \exp(-u[x, t]) \\ -\eta_\omega[x] \exp(-u[x, t])\omega_{yy}[y, t] \exp(-u[y, t])/K \tag{59}$$

$$J_u[x, y] = \exp(-u[x, t]) J_\omega[x, y] \exp(-u[y, t]) \tag{60}$$

where K is the deformation of the Hamiltonian H along the symmetry η_u, $J_\omega[x, y]$ is a Hamiltonian operator for heat equation (14). We used symmetry (57).

Furthermore, note the following relationships:

$$H[\omega] = \int \omega[x, t]\,\mathrm{d}x \Rightarrow H[u] = \int \exp(u[x, t])\,\mathrm{d}x \tag{61}$$

$$\mathcal{L}_{\eta_\omega} H[\omega] = \int \frac{\delta H[\omega]}{\delta\omega[x, t]}\eta_\omega\,\mathrm{d}x = \int \eta_\omega = K \tag{62}$$

$$\mathcal{L}_{\eta_u} H[u] = \int \frac{\delta H[u]}{\delta u[x, t]}\eta_u\,\mathrm{d}x = \int \exp(u[x, t])\eta_u = \int \eta_\omega = K. \tag{63}$$

Expression (60) is not altered if we include the terms that raise the rank of the Poisson-bracket operator $J_\infty[u] = \exp(u[x, t]) J_\infty[\omega] \exp(u[y, t])$. Similarly, for the Burger equation we have

$$J_v[x, y] = -D \exp(-u[x, t]) J_\omega[x, y] \exp(-u[y, t]) D. \tag{64}$$

7. Other examples

Consider the Krichever–Novikov (KN) equation [13, 14] for the field $\psi[x, t]$

$$\psi_t[x, t] = \psi_{xxx}[x, t] - \frac{3}{2}\frac{\psi_{xx}^2[x, t]}{\psi_x[x, t]} \equiv f_{(1)}. \tag{65}$$

It is not difficult to see that the following quantities are conserved:

$$H^{(1)} = \int \frac{\mathrm{d}x}{\psi_1[x, t]} \tag{66}$$

$$H^{(2)} = \int \frac{\psi[x, t]}{\psi_1[x, t]}\,\mathrm{d}x. \tag{67}$$

Considering the symmetry vectors, $\psi[x, t]$ and $\psi^2[x, t]$, and the constant of motion $H^{(1)}$ as a Hamiltonian gives the following Poisson-bracket structures:

$$J_{(1)}^{(\psi)}[x, y] = \frac{1}{-H^{(1)}}(f_{(1)}[x]\psi[y] - f_{(1)}[y]\psi[x]) \tag{68}$$

$$J_{(2)}^{(\psi)}[x, y] = \frac{1}{-2H^{(2)}}(f_{(1)}[x]\psi^2[y] - f_{(1)}[y]\psi^2[x]). \tag{69}$$

Using the operator $M = -(D + \frac{\psi_2[x,t]}{\psi_1[x,t]})D\frac{1}{2\psi_1[x,t]}D$, we can then find the Poisson-bracket structures for the Korteweg–deVries (KdV) equation

$$\mathrm{KdV}[u[x, t]] \Rightarrow u_t[x, t] = u_{xxx}[x, t] + 6u[x, t]u_x[x, t] \tag{70}$$

$$J_{(i)}^{(u)}[x, y] = M[x] J_{(i)}^{(\psi)}[x, y] M^*[y]. \tag{71}$$

Note that these Poisson-bracket structures for the KdV equation are non-local. What we have indirectly demonstrated is that, by means of Miura-type coordinate changes, the non-local Poisson structures for KdV (71), can become local. Basically we have introduced the pseudo-potential $\psi[x, t]$ [15] for which this structure becomes local.

8. Conclusions

The method presented here may constitute an alternative to constructing a Hamiltonian description of the system (1) without using a Lagrangian. We obtain infinite-range Poisson-bracket structures taking advantage of the existence of an infinite group of symmetries, given by the use of the Lenard-type recursion operator, when it exists.

The procedure outlined in the paper to raise the rank of the Poisson-bracket structure may be viewed as the 'inverse' of the Dirac method. In fact, the Dirac method lowers the rank of the Poisson brackets to turn them into Dirac brackets in which second-class constraints commute with any function of the dynamical variables (they are Casimir functions for the Dirac brackets), while this procedure raises the rank of the Poisson bracket operators thus reducing the number of Casimir functions.

Once we have a Poisson-bracket operator for one equation, using a Miura-type transformation we can obtain similar structures for other equations. The application of this method to other hierarchies of equations that possess Lenard-type recursion operators similar to KdV hierarchies, is direct and will be published elsewhere.

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